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COEFFICIENT BOUNDS FOR REGULAR AND BI-UNIVALENT FUNCTIONS LINKED WITH GEGENBAUER POLYNOMIALS

Abstract. The main goal of the paper is to initiate and explore two sets of regular and bi-univalent (or bi-Schlicht) functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ linked with Gegenbauer polynomials. We investigate certain coefficient bounds for functions in these families. Continuing the study on the initial coefficients of these families, we obtain the functional of Fekete-Szegö for each of the two families. Furthermore, we present few interesting observations of the results investigated.

Key words: *Fekete-Szegö functional, regular function, bi-univalent function, Gegenbauer polynomials*

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1. Preliminaries. Let the set of complex numbers be denoted by $\mathbb{C}$, the set of normalized regular functions in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ that have the power series of the form

$$g(z) = z + d_2 z^2 + d_3 z^3 + \ldots = z + \sum_{j=2}^{\infty} d_j z^j, \qquad (1)$$

be denoted by $\mathcal{A}$ and the set of all functions of $\mathcal{A}$ that are univalent in $\mathfrak{D}$ be denoted by $\mathcal{S}$. The famous Koebe theorem (see [5]) ensures that any function $g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$ and $r_0(g) \geqslant 1/4$, $z, \omega \in \mathfrak{D}$, where

$$g^{-1}(\omega) = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2 d_3 + d_4)\omega^4 + \ldots \qquad (2)$$

A function g of $\mathcal{A}$ is said to be bi-univalent (or bi-schlicht) in $\mathfrak{D}$ if g and its inverse g^{-1} are both univalent (or schlicht) in $\mathfrak{D}$. The set of bi-univalent functions having the form (1) is indicated by Σ. Investigations of the family Σ begun five decades ago by Lewin [9] and Brannan and Clunie [4]. Later, Tan [11] found the initial coefficient bounds of bi-univalent functions. Moreover, Brannan and Taha [3] presented and investigated certain subsets of Σ similar to convex and starlike functions of order σ $(0 \leqslant \sigma < 1)$ in $\mathfrak{D}$. Some interesting results concerning initial bounds for certain special sets of Σ have appeared: [7] and [10].

Let the set of real numbers be $\mathbb{R} = (-\infty, \infty)$ and the set of positive integers be $\mathbb{N} := \mathbb{N}_0 \backslash \{0\} = \{1, 2, 3, \ldots\}$.

Recently, Kiepiela et al. [8] have examined the Gegenbauer polynomials (or ultraspherical polynomials) $C_j^{\alpha}(x)$. They are orthogonal polynomials on [-1, 1] that can be defined by the recurrence relation

$$C_j^{\alpha}(x) = \frac{2x(j+\alpha-1)C_{j-1}^{\alpha}(x) - (j+2\alpha-2)C_{j-2}^{\alpha}(x)}{j}, \tag{3}$$
$$C_0^{\alpha}(x) = 1\,,\; C_1^{\alpha}(x) = 2\alpha x$$

where $j \in \mathbb{N}\backslash\{1\}$. It is easy to see from (3) that $C_2^{\alpha}(x) = 2\alpha(1+\alpha)x^2 - \alpha$. For $\alpha \in \mathbb{R}\backslash\{0\}$, a generating function of the sequence $C_j^{\alpha}(x)$, $j \in \mathbb{N}$, is defined by (see [1]):

$$\mathcal{H}_{\alpha}(x,z) := \sum_{j=0}^{\infty} C_j^{\alpha}(x) z^j = \frac{1}{(1-2xz+z^2)^{\alpha}}, \tag{4}$$

where $z \in \mathfrak{D}$ and $x \in [-1, 1]$.

Two particular cases of $C_j^{\alpha}(x)$ are $i)$ $C_j^{1}(x)$: the Chebyshev polynomials of the second kind and $ii)$ $C_j^{\frac{1}{2}}(x)$: the Legendre polynomials (see [2]).

In the literature, the estimates on $|d_2|$, $|d_3|$ and the famous inequality of Fekete-Szegö were determined for bi-univalent functions linked with certain polynomials like (p, q)-Lucas polynomials, second kind Chebyshev polynomials, Horadam polynomials and Gegenbauer polynomials. It is well-known that these polynomials and other special polynomials play a potentially important role in the approximation theory, statistical, physical, mathematical, and engineering sciences.

The recent research trend is the study of bi-univalent functions linked with any of the above mentioned polynomials. However, there has been little work done on bi-univalent functions linked with Gegenbauer polynomials. To initiate and explore the study on bi-univalent functions linked with Gegenbauer polynomials, we present two special families of Σ subordinate to Gegenbauer polynomials $C_j^{\alpha}(x)$ as in (3) with the generating function (4).

For regular functions g and f in $\mathfrak{D}$, g is said to subordinate to f , if there is a Schwarz function ψ in $\mathfrak{D}$, such that $\psi(0) = 0$, $|\psi(z)| < 1$, and $g(z) = f(\psi(z)), z \in \mathfrak{D}$. This subordination is indicated as $g \prec f$ or $g(z) \prec f(z)$. Specifically, when $f \in \mathcal{S}$ in $\mathfrak{D}$, then $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$.

Inspired by the recent articles and the new trends on functions $\in \Sigma$, we present two families of Σ associated with Gegenbauer polynomials $C_j^{\alpha}(x)$ as in (3) with the generating function (4).

Throughout this paper, an inverse function $g^{-1}(\omega) = f(\omega)$ is as in (2) and $\mathcal{H}_{\alpha}(x, z)$ is as in (4).

Definition 1. *A function g in Σ having the power series (1) is said to be in the family $S\mathfrak{S}_{\Sigma}^{\alpha}(x, \gamma, \mu)$, $0 \leqslant \gamma \leqslant 1$, $\mu \geqslant 0$, $1/2 < x \leqslant 1$, and α a nonzero real constant, if*

$$\frac{zg'(z) + \mu z^2 g''(z)}{\gamma g(z) + (1-\gamma)z} \prec \mathcal{H}_{\alpha}(x, z), \quad z \in \mathfrak{D}$$

and

$$\frac{\omega f'(\omega) + \mu \omega^2 f''(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec \mathcal{H}_{\alpha}(x, \omega), \quad \omega \in \mathfrak{D}.$$

The above defined family $\mathrm{S}\mathfrak{S}_{\Sigma}^{\alpha}(x, \gamma, \mu)$ is of special interest, for it contains new subfamilies of Σ for particular values of γ and μ, as illustrated below:

1. $SK_{\Sigma}^{\alpha}(x, \gamma) \equiv S\mathfrak{S}_{\Sigma}^{\alpha}(x, \gamma, 0)$ is the set of functions $g \in \Sigma$ satisfying

$$\frac{zg'(z)}{\gamma g(z) + (1-\gamma)z} \prec \mathcal{H}_{\alpha}(x, z), \quad \frac{\omega f'(\omega)}{\gamma f(\omega) + (1-\gamma)\omega} \prec \mathcal{H}_{\alpha}(x, \omega),\ z, \omega \in \mathfrak{D}.$$

2. $SL_{\Sigma}^{\alpha}(x, \mu) \equiv S\mathfrak{S}_{\Sigma}^{\alpha}(x, 0, \mu)$ is the family of functions $g \in \Sigma$ satisfying

$$g'(z) + \mu z g''(z) \prec \mathcal{H}_{\alpha}(x, z), \quad f'(\omega) + \mu \omega f''(\omega) \prec \mathcal{H}_{\alpha}(x, \omega),\ z, \omega \in \mathfrak{D}.$$

3. $SM_{\Sigma}^{\alpha}(x, \mu) \equiv S\mathfrak{S}_{\Sigma}^{\alpha}(x, 1, \mu)$ is the class of functions $g \in \Sigma$ satisfying

$$\left(\frac{zg'(z)}{g(z)}\right) + \mu \left(\frac{zg''(z)}{g(z)}\right) \prec \mathcal{H}_{\alpha}(x, z), \quad z \in \mathfrak{D}$$

and
$$\left(\frac{\omega f'(\omega)}{f(\omega)}\right)+\mu\left(\frac{\omega f''(\omega)}{f(\omega)}\right)\prec \mathcal{H}_\alpha(x,\omega),\quad \omega\in\mathfrak{D}.$$

Definition 2. *A function $g\in\Sigma$ having the power series (1) is said to be in the family $S\mathfrak{B}_\Sigma^\alpha(x,\gamma,\tau)$, $0\leqslant\gamma\leqslant 1$, $\tau\geqslant 1$, $1/2<x\leqslant 1$, and α a nonzero real constant, if*
$$\frac{z[g'(z)]^\tau}{\gamma g(z)+(1-\gamma)z}\prec \mathcal{H}_\alpha(x,z),\quad z\in\mathfrak{D}$$
and
$$\frac{\omega[f'(\omega)]^\tau}{\gamma f(\omega)+(1-\gamma)\omega}\prec \mathcal{G}(x,\omega),\quad \omega\in\mathfrak{D}.$$

Note that the certain choices of γ lead the family $S\mathfrak{B}_\Sigma^\alpha(x,\gamma,\tau)$ to the following two subclasses:

1. $SP_\Sigma^\alpha(x,\tau)\equiv S\mathfrak{B}_\Sigma^\alpha(x,0,\tau)$ is the set of functions $g\in\Sigma$ satisfying
$$[g'(z)]^\tau\prec \mathcal{H}_\alpha(x,z),\quad z\in\mathfrak{D}\quad\text{and}\quad [f'(\omega)]^\tau\prec \mathcal{H}_\alpha(x,\omega),\quad \omega\in\mathfrak{D},$$
2. $S\mathfrak{N}_\Sigma^\alpha(x,\tau)\equiv S\mathfrak{B}_\Sigma^\alpha(x,1,\tau)$ is the class of functions $g\in\Sigma$ satisfying
$$\frac{z[g'(z)]^\tau}{g(z)}\prec \mathcal{H}_\alpha(x,z),\quad z\in\mathfrak{D}\quad\text{and}\quad \frac{\omega[f'(\omega)]^\tau}{f(\omega)}\prec \mathcal{H}_\alpha(x,\omega),\quad \omega\in\mathfrak{D}.$$

Remark 1. *Note that*

i) $S\mathfrak{B}_\Sigma^\alpha(x,\gamma,1)\equiv SK_\Sigma^\alpha(x,\gamma)$,

ii) $S\mathfrak{N}_\Sigma^\alpha(x,1)\equiv SK_\Sigma^\alpha(x,1)\equiv SM_\Sigma^\alpha(x,0)$,

iii) $SP_\Sigma^\alpha(x,1)\equiv SK_\Sigma^\alpha(x,0)\equiv SL_\Sigma^\alpha(x,0)$.

In Section 2, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegö [6] for functions of the form (1) in $S\mathfrak{S}_\Sigma^\alpha(x,\gamma,\mu)$. Interesting consequences of our result are also presented. In Section 3, we derive the estimates for $|d_2|$, $|d_3|$ and the inequality of Fekete-Szegö for functions of the form (1) in $S\mathfrak{B}_\Sigma^\alpha(x,\gamma,\tau)$. A few interesting consequences of the result are mentioned.

2. Estimates for the function family $S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$. In the following theorem, we determine the initial coefficient bounds and the inequality of Fekete-Szegö for functions in $S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$.

Theorem 1. *If the function $g \in S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$, then*

$$|d_2| \leqslant \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2(\mu+1)-\gamma)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1)+1)\alpha x^2|}}, \quad (5)$$

$$|d_3| \leqslant \frac{4\alpha^2x^2}{(2(\mu+1)-\gamma)^2} + \frac{2|\alpha|x}{(3(2\mu+1)-\gamma)} \quad (6)$$

and, for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant$$
$$\leqslant \begin{cases} \dfrac{2|\alpha|x}{(3(2\mu+1)-\gamma)}, & |1-\delta| \leqslant \mathfrak{J}, \\ \dfrac{8\alpha^2x^3\,|1-\delta|}{|(2(\mu+1)-\gamma)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1)+1)\alpha x^2|}, & |1-\delta| \geqslant \mathfrak{J}, \end{cases} \quad (7)$$

where

$$\mathfrak{J} = \left|\frac{(2(\mu+1)-\gamma)^2(1-2x^2)+2((\gamma-1)^2-4\mu(\mu-1)+1)\alpha x^2}{4(3(2\mu+1)-\gamma)\alpha x^2}\right|. \quad (8)$$

Proof. Let $g \in S\mathfrak{S}_{\Sigma}^{\alpha}(x,\gamma,\mu)$. Then, for two regular functions $\mathfrak{M}$, $\mathfrak{N}$ with $\mathfrak{M}(0)=0$, $|\mathfrak{M}(z)|<1$, $\mathfrak{N}(0)=0$ and $|\mathfrak{N}(\omega)|<1$, $z,\omega \in \mathfrak{D}$, and on account of Definition 1, we can write

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma g(z)+(1-\gamma)z} = \mathcal{H}_{\alpha}(x,\mathfrak{M}(z)),$$

$$\frac{\omega f'(\omega)+\mu\omega^2f''(\omega)}{\gamma f(\omega)+(1-\gamma)\omega} = \mathcal{H}_{\alpha}(x,\mathfrak{N}(\omega)),$$

or, equivalently,

$$\frac{zg'(z)+\mu z^2g''(z)}{\gamma g(z)+(1-\gamma)z} = 1+C_1^{\alpha}(x)+C_2^{\alpha}(x)\mathfrak{m}(z)+C_3^{\alpha}(x)(\mathfrak{m}(z))^2+\ldots, \quad (9)$$

$$\frac{\omega f'(\omega)+\mu\omega^2f''(\omega)}{\gamma f(\omega)+(1-\gamma)\omega} = 1+C_1^{\alpha}(x)+C_2^{\alpha}(x)\mathfrak{n}(\omega)+C_3^{\alpha}(x)(\mathfrak{n}(\omega))^2+\ldots \quad (10)$$

From (9) and (10), in view of (3), we find

$$\frac{zg'(z)+\mu z^2 g''(z)}{\gamma g(z)+(1-\gamma)z} = 1 + C_1^\alpha(x)\mathfrak{m}_1 z + [C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2]z^2 + \ldots, \quad (11)$$

$$\frac{\omega f'(\omega)+\mu\omega^2 f''(\omega)}{\gamma f(\omega)+(1-\gamma)\omega} = 1 + C_1^\alpha(x)\mathfrak{n}_1\omega + [C_1^\alpha(x)\mathfrak{n}_2 + C_1^\alpha(x)\mathfrak{n}_1^2]\omega^2 + \ldots \quad (12)$$

It is well known that if $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \ldots| < 1$, $z \in \mathfrak{D}$, and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + \ldots| < 1$, $\omega \in \mathfrak{D}$, then

$$|\mathfrak{m}_i| \leqslant 1 \text{ and } |\mathfrak{n}_i| \leqslant 1 \, (i \in \mathbb{N}). \quad (13)$$

We easily get the following by equating the corresponding coefficients in (11) and (12):

$$(2(\mu+1)-\gamma)d_2 = C_1^\alpha(x)\mathfrak{m}_1, \quad (14)$$

$$(3(2\mu+1)-\gamma)d_3 - (2(\mu+1)-\gamma)\gamma\, d_2^2 = C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2, \quad (15)$$

$$-\,(2(\mu+1)-\gamma)\, d_2 = C_1^\alpha(x)\mathfrak{n}_1, \quad (16)$$

$$-(3(2\mu+1)-\gamma)d_3 + (\gamma^2 - 2(\mu+2)\gamma + 6(2\mu+1))d_2^2 = C_1^\alpha(x)\mathfrak{n}_2 + C_2^\alpha(x)\mathfrak{n}_1^2. \quad (17)$$

It follows easily from (14) and (16) that

$$\mathfrak{m}_1 = -\mathfrak{n}_1 \quad (18)$$

$$2(2(\mu+1)-\gamma)^2 d_2^2 = (\mathfrak{m}_1^2 + \mathfrak{n}_1^2)(C_1^\alpha(x))^2. \quad (19)$$

If we add (15) and (17), we obtain

$$2(\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1))d_2^2 = C_1^\alpha(x)(\mathfrak{m}_2 + \mathfrak{n}_2) + C_2^\alpha(x)(\mathfrak{m}_1^2 + \mathfrak{n}_1^2). \quad (20)$$

Substituting the value of $\mathfrak{m}_1^2 + \mathfrak{n}_1^2$ from (19) in (20), we get

$$d_2^2 = \frac{(C_1^\alpha(x))^3(\mathfrak{m}_2+\mathfrak{n}_2)}{2\left[(\gamma^2 - (2\mu+3)\gamma + 3(2\mu+1))(C_1^\alpha(x))^2 - (2(\mu+1)-\gamma)^2 C_2^\alpha(x)\right]}, \quad (21)$$

which yields (5) on using (13).

After subtracting (17) from (15) and then using (18), we obtain

$$d_3 = d_2^2 + \frac{C_1^\alpha(x)(\mathfrak{m}_2 - \mathfrak{n}_2)}{2(3(2\mu+1)-\gamma)}. \quad (22)$$

Then, in view of (19), equation (22) becomes

$$d_3 = \frac{(C_1^\alpha(x))^2(\mathfrak{m}_1^2+\mathfrak{n}_1^2)}{2(2(\mu+1)-\gamma)^2} + \frac{C_1^\alpha(x)(\mathfrak{m}_2-\mathfrak{n}_2)}{2(3(2\mu+1)-\gamma)},$$

which yields (6) on applying (13).

From (21) and (22), for $\delta \in \mathbb{R}$ we get

$$|d_3-\delta d_2^2| = |C_1^\alpha(x)|\Big|\Big(T(\delta,x)+\frac{1}{2(3(2\mu+1)-\gamma)}\Big)\mathfrak{m}_2+ \\ +\Big(T(\delta,x)-\frac{1}{2(3(2\mu+1)-\gamma)}\Big)\mathfrak{n}_2\Big|,$$

where

$$T(\delta,x) = \frac{(1-\delta)\,(C_1^\alpha(x))^2}{2\,[(\gamma^2-(2\mu+3)\gamma+3(2\mu+1))(C_1^\alpha(x))^2-(2(\mu+1)-\gamma)^2C_2^\alpha(x)]}.$$

In view of (3), we conclude that

$$|d_3-\delta d_2^2| \leqslant \\ \leqslant \begin{cases} \dfrac{|C_1^\alpha(x)|}{(3(2\mu+1)-\gamma)}, & 0\leqslant |T(\delta,x)| \leqslant \dfrac{1}{2(3(2\mu+1)-\gamma)}, \\ 2|C_1^\alpha(x)||T(\delta,x)|, & |T(\delta,x)| \geqslant \dfrac{1}{2(3(2\mu+1)-\gamma)}, \end{cases}$$

which gives (7) with $\mathfrak{J}$ as in (8). Thus, the proof of Theorem 1 is completed. □

Setting $\mu = 0$ in Theorem 1, we obtain

Corollary 1. *If the function $g \in SK_\Sigma^\alpha(x,\gamma)$, then*

$$|d_2| \leqslant \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2-\gamma)^2(1-2x^2)+2((\gamma-1)^2+1)\alpha x^2|}},$$

$$|d_3| \leqslant \frac{4\alpha^2x^2}{(2-\gamma)^2}+\frac{2|\alpha|x}{3-\gamma}$$

and for some $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leqslant \begin{cases} \dfrac{2|\alpha|x}{(3-\gamma)}, & |1-\delta| \leqslant \mathfrak{G}, \\ \dfrac{8\alpha^2x^2\ |1-\delta|}{|(2-\gamma)^2(1-2x)^2+2((\gamma-1)^2+1)\alpha x^2|}, & |1-\delta| \geqslant \mathfrak{G}, \end{cases}$$

where

$$\mathfrak{G} = \left|\frac{(2-\gamma)^2(1-2x^2)+2((\gamma-1)^2+1)\alpha x^2}{4(3-\gamma)\alpha x^2}\right|.$$

Remark 2. *Corollary 1 reduces to Corollary 8 and Corollary 9 of Amourah et al. [2], when $\gamma = 1$ and $\gamma = 0$, respectively.*

Letting $\gamma = 0$ in Theorem 1, we have

Corollary 2. *If the function $g \in SL_{\Sigma}^{\alpha}(x,\mu)$, then*

$$|d_2| \leqslant \frac{|\alpha|x\sqrt{2x}}{\sqrt{|(\mu+1)^2(1-2x^2)-(2\mu(\mu-1)-1)\alpha x^2|}},$$

$$|d_3| \leqslant \frac{\alpha^2x^2}{(\mu+1)^2} + \frac{2|\alpha|x}{3(2\mu+1)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leqslant \begin{cases} \dfrac{2|\alpha|x}{3(2\mu+1)}, & |1-\delta| \leqslant \mathfrak{G}_1, \\ \dfrac{2\alpha^2x^3\ |1-\delta|}{|(\mu+1)^2(1-2x^2)-(2\mu(\mu-1)-1)\alpha x^2|}, & |1-\delta| \geqslant \mathfrak{G}_1, \end{cases}$$

where

$$\mathfrak{G}_1 = \left|\frac{(\mu+1)^2(1-2x^2)-(2\mu(\mu-1)-1)\alpha x^2}{3(2\mu+1)\alpha x^2}\right|.$$

Remark 3. *For $\mu = 0$, Corollary 2 coincides with Corollary 9 of [2].*

Allowing $\gamma = 1$ in Theorem 1, we get

Corollary 3. *If the function $g \in SM_{\Sigma}^{\alpha}(x,\mu)$, then*

$$|d_2| \leqslant \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2\mu+1)^2(1-2x^2)-2(4\mu(\mu-1)-1)\alpha x^2|}},$$

$$|d_3| \leqslant \frac{4\alpha^2x^2}{(2\mu+1)^2} + \frac{|\alpha|x}{(3\mu+1)}$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leqslant \begin{cases} \dfrac{|\alpha|x}{(3\mu+1)}, & |1-\delta| \leqslant \mathfrak{J}_1 \\ \dfrac{8\alpha^2x^3\ |1-\delta|}{|(2\mu+1)^2(1-2x^2)-2(4\mu(\mu-1)-1)\alpha x^2]|}, & |1-\delta| \geqslant \mathfrak{J}_1, \end{cases}$$

where $\mathfrak{J}_1 = \left|\dfrac{(2\mu+1)^2(1-2x^2)-2(4\mu(\mu-1)-1)\alpha x^2}{8(3\mu+1)\alpha x^2}\right|$.

Remark 4. *We obtain Corollary 8 of [2] from Corollary 3, when $\mu = 0$ (Also see [1]).*

3. Estimates for the function family $S\mathfrak{B}^{\alpha}_{\sum}(x, \gamma, \tau)$. In the next theorem, we find the first two Taylor-Maclaurin coefficients and the inequality of Fekete-Szegö for functions in $S\mathfrak{B}^{\alpha}_{\Sigma}(x, \gamma, \tau)$.

Theorem 2. *If the function $g \in S\mathfrak{B}^{\alpha}_{\Sigma}(x, \gamma, \tau)$, then*

$$|d_2| \leqslant \frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2\tau-\gamma)^2(1-2x^2)+2(\gamma^2+2(\tau-\gamma))\alpha x^2|}}, \tag{23}$$

$$|d_3| \leqslant \frac{4(\alpha x)^2}{(2\tau-\gamma)^2} + \frac{2|\alpha|x}{(3\tau-\gamma)} \tag{24}$$

and for $\delta \in \mathbb{R}$:

$$|d_3 - \delta d_2^2| \leqslant \begin{cases} \dfrac{2|\alpha|x}{(3\tau-\gamma)}, & |1-\delta| \leqslant \Omega, \\ \dfrac{|1-\delta|\, 8\alpha^2x^3}{|(2\tau-\gamma)^2(1-2x^2)+2(\gamma^2+2(\tau-\gamma))\alpha x^2|}, & |1-\delta| \geqslant \Omega, \end{cases} \tag{25}$$

where

$$\Omega = \left|\frac{(2\tau-\gamma)^2(1-2x^2)+2(\gamma^2+2(\tau-\gamma))\alpha x^2}{4(3\tau-\gamma)\alpha x^2}\right|.$$

Proof. Let $g \in S\mathfrak{B}^{\alpha}_{\Sigma}(x, \gamma, \tau)$. Then, for some regular functions $\mathfrak{M}$ and $\mathfrak{N}$ such that $\mathfrak{M}(0) = 0$, $|\mathfrak{M}(z)| = |\mathfrak{m}_1 z + \mathfrak{m}_2 z^2 + \mathfrak{m}_3 z^3 + \ldots| < 1, \mathfrak{N}(0) = 0$ and $|\mathfrak{N}(\omega)| = |\mathfrak{n}_1\omega + \mathfrak{n}_2\omega^2 + \mathfrak{n}_3\omega^3 + \ldots| < 1$, z, $\omega \in \mathfrak{D}$, and on account of Definition 2, we can write

$$\frac{z[g'(z)]^\tau}{\gamma g(z) + (1-\gamma)z} = \mathcal{H}_\alpha(x, \mathfrak{M}(z)), \quad z \in \mathfrak{D},$$

$$\frac{\omega[f'(\omega)]^\tau}{\gamma f(\omega) + (1-\gamma)\omega} = \mathcal{H}_\alpha(x, \mathfrak{N}(\omega)), \quad \omega \in \mathfrak{D}.$$

Following the procedure similar to the proof of Theorem 1, one gets

$$(2\tau-\gamma)d_2 = C_1^\alpha(x)\mathfrak{m}_1, \tag{26}$$

$$(\gamma^2 - 2\tau\gamma + 2\tau(\tau-1))d_2^2 + (3\tau-\gamma)d_3 = C_1^\alpha(x)\mathfrak{m}_2 + C_2^\alpha(x)\mathfrak{m}_1^2, \tag{27}$$

$$-(2\tau-\gamma)d_2 = H_2(x)\mathfrak{n}_1, \tag{28}$$

$$(\gamma^2-2(\tau+1)\gamma+2\tau(\tau+2))d_2^2-(3\tau-\gamma)d_3 = C_1^\alpha(x)\mathfrak{n}_2+C_2^\alpha(x)\mathfrak{n}_1^2. \tag{29}$$

The results (23)–(25) now follow from (26)–(29) by adopting the procedure as in Theorem 1. $\square$

Putting $\gamma=0$ in Theorem 2, we get

Corollary 1. *If the function $g\in SP_\Sigma^\alpha(x,\tau)$, then*

$$|d_2|\leqslant\frac{|\alpha|x\sqrt{2x}}{\sqrt{|\tau^2(1-2x^2)+\tau\alpha x^2|}},\quad |d_3|\leqslant\frac{\alpha^2x^2}{\tau^2}+\frac{2|\alpha|x}{3\tau}$$

and for some $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2|\leqslant\begin{cases}\dfrac{2|\alpha|x}{3\tau}, & |1-\delta|\leqslant\left|\dfrac{\tau(1-2x^2)+\alpha x^2}{3\alpha x^2}\right|,\\[2ex] \dfrac{|1-\delta|\,8\alpha^2x^3}{|4\tau^2(1-2x^2)+4\tau\alpha x^2|}, & |1-\delta|\geqslant\left|\dfrac{\tau(1-2x^2)+\alpha x^2}{3\alpha x^2}\right|.\end{cases}$$

Remark 5. *Corollary 1 coincides with [2, Corollary 9], when $\tau=1$.*

Taking $\gamma=1$ in Theorem 2, we get

Corollary 2. *If the function $g\in S\mathfrak{N}_\Sigma^\alpha(x,\tau)$, then*

$$|d_2|\leqslant\frac{2|\alpha|x\sqrt{2x}}{\sqrt{|(2\tau-1^2)(1-2x^2)+2(2\tau-1)\alpha x^2|}},\quad |d_3|\leqslant\frac{4\alpha^2x^2}{(2\tau-1)^2}+\frac{2|\alpha|x}{(3\tau-1)}$$

and for $\delta\in\mathbb{R}$:

$$|d_3-\delta d_2^2|\leqslant\begin{cases}\dfrac{2|\alpha|x}{(3\tau-1)}, & |1-\delta|\leqslant\mathfrak{G}_2,\\[2ex] \dfrac{|1-\delta|\,8\alpha^2x^3}{|(2\tau-1)^2(1-2x^2)+2(2\tau-1)\alpha x^2|}, & |1-\delta|\geqslant\mathfrak{G}_2,\end{cases}$$

where

$$\mathfrak{G}_2=\left|\frac{(2\tau-1)^2(1-2x^2)+2(2\tau-1)\alpha x^2}{4(3\tau-1)\alpha x^2}\right|.$$

Remark 6. *Corollary 2 reduces to Corollary 8 of [2], when $\tau=1$.*

4. Conclusion. Two special families of regular and bi-univalent (or bi-schlicht) functions linked with Gegenbauer polynomials are introduced and explored. Bounds of the first two coefficients $|d_2|, |d_3|$ and the celebrated Fekete- Szegö functional have been fixed for each of the two families. Through corollaries of our main results, we have highlighted many interesting new consequences.

The special families examined in this research paper and linked with Gegenbauer polynomials could inspire further research related to other aspects, such as families using q-derivative operator, q-integral operator, meromorphic bi-univalent function families associated with Al-Oboudi differential operator, and families that use integro-differential operators.

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