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A note on double Laplace decomposition method and nonlinear partial differential equations

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Abstract: In this article we propose a new technique, namely double Laplace decomposition method for solving nonlinear partial differential equation. The technique is described and illustrated with some examples.

Keywords: Double Laplace transform, inverse laplace transform, nonlinear partial differential equation.

1 Introduction

The linear and nonlinear problems , that are appear in many areas of scientific research such as solid state physics, wave equation, telegraph equation, plasma physics, fluid mechanics, which are modeled by linear and nonlinear partial differential equations. Also the double Laplace transform and some of its application are used to solve general linear telegraph equation and wave equation with initial and boundary conditions see [3]. Also double Laplace transform applied by Eltayeb and Kilicman [4] and [5] to solved non-homogeneous wave equation with variable coefficients. In this work we use the double Laplace decomposition methods to solve nonlinear partial differential equation. In special cases four examples are given. We are recalling the following definitions which are given by [2]. the double Laplace transform defined as

$$L_xL_t\left[f(x,s)\right] = F(p,s) = \int_0^\infty e^{-px}\int_0^\infty e^{-st} f(x,t)dt\,dx \tag{1}$$

where $x,t>0$ and p,s complex value and further double Laplace transform of the first order partial derivatives are given by

$$L_xL_t\left[\frac{\partial f(x,t)}{\partial x}\right] = pF(p,s)-F(0,s).$$

Similarly the double Laplace transform for second partial derivative with respect to x and t are defined as follows

$$L_{xx}\left[\frac{\partial^2 f(x,t)}{\partial^2 x}\right] = p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}$$
$$L_{tt}\left[\frac{\partial^2 f(x,t)}{\partial^2 t}\right] = s^2F(p,s)-sF(p,0)-\frac{\partial F(p,0)}{\partial t}$$

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Theorem 1. *Let $f(x,t)$ and $g(x,t)$ be having double Laplace transform. Then double Laplace transform of the double convolution of the $f(x,t)$ and $g(x,t)$,*

$$f(x,t)**g(x,t)=\int_0^t\int_0^x f(x-\eta,t-\zeta)g(\eta,\zeta)d\eta d\zeta$$

is given by

$$L_xL_t\left[f(x,s)**g(x,t);p,s\right]=F(p,s)G(p,s)$$

Proof. By applying the definition of double Laplace transform and double convolutions, we get

$$\begin{aligned}L_xL_t\left[f(x,s)**g(x,t);p,s\right]&=\int_0^\infty\int_0^\infty e^{-px-st}\left(f(x,t)**g(x,t)\right)dt\,dx\\&=\int_0^\infty\int_0^\infty e^{-px-st}\left(\int_0^t\int_0^x f(x-\eta,t-\zeta)g(\eta,\zeta)d\eta d\zeta\right)dx\,dt,\end{aligned}$$

let $\alpha=x-\eta$ and $\beta=t-\zeta$, and the region of integration becomes as $\eta\geq 0$, $\zeta\geq 0$ and $\alpha\geq 0$, $\beta\geq 0$, we obtain

$$L_xL_t\left[f(x,s)**g(x,t);p,s\right]=\left(\int_0^\infty\int_0^\infty e^{-p\eta-s\zeta}f(\eta,\zeta)d\eta d\zeta\right)\left(\int_0^\infty\int_0^\infty e^{-p\alpha-s\beta}g(\alpha,\beta)d\alpha d\beta\right).$$

Then, one can see that

$$L_xL_t\left[f(x,s)**g(x,t);p,s\right]=F(p,s)G(p,s)$$

In the next theorem we study the nonlinear partial differential equation with convolution operator by using double Laplace decomposition methods

Theorem 2. *Consider the nonlinear partial differential equation with convolution term as follows*

$$\frac{\partial^2 u(x,t)}{\partial^2 x}+Ru(x,t)+Ku(x,t)=g(x,t)**h(x,t) \tag{2}$$

$$u(0,t)=f_1(t),\frac{\partial u(0,t)}{\partial x}=f_2(t) \tag{3}$$

R represents a linear operator and K denoted by nonlinear differential operator with convolution as follows

$$Ku(x,t)=(u(x,t))^n**\left(\frac{\partial^2 u(x,t)}{\partial^2 t}\right)^m,$$

then the solution of Eq(2) given by

$$\left(\sum_{n=0}^{\infty}u_n(x,t)\right)=f_1(t)+xf_2(t)-L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[R\sum_{n=0}^{\infty}u_n(x,t)+\sum_{n=0}^{\infty}A_n\right]\right]+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[g(x,t)**h(x,t)\right]\right] \tag{4}$$

where $\sum_{n=0}^{\infty}A_n=Ku(x,t)$.

Proof. By taking double Laplace transform for both sides of Eq(2) and single Laplace transform for Eq(3) we obtain

$$U(p,s)=\frac{F_1(s)}{p}+\frac{F_2(s)}{p^2}-\frac{1}{p^2}L_xL_t\left[Ru(x,t)+Ku(x,t)\right]+\frac{1}{p^2}L_xL_t\left[\left[g(x,t)**h(x,t)\right]\right]. \tag{5}$$

On using double inverse Laplace transform for Eq(5) we have

$$u(x,t)=f_1(t)+xf_2(t)-L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[Ru(x,t)+Ku(x,t)\right]\right]+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[g(x,t)**h(x,t)\right]\right], \tag{6}$$

applying the decomposition method , then we consider the solution as an infinite series given as follows

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{7}$$

The nonlinear operator is decompose as follows

$$Ku(x,t)=\sum_{n=0}^{\infty}A_n \tag{8}$$

Where A_n are Adomian polynomials given by

$$A_n=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i u_i\right)\right]_{\lambda=0}, \quad n=0,1,2,.... \tag{9}$$

By subsitituting Eq(7) and Eq(8) into Eq(6) we complete the proof.

The purpose of this parts is to study the use of modified double Laplace transform algorithm for the nonlinear partial differential equations. We consider the general form of second order nonhomogeneous nonlinear partial differential equations with initial conditions is given below

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=h(x,t) \tag{10}$$

with initial condition

$$u(0,t)=f(t), \quad u_x(0,t)=g(t), \tag{11}$$

where L denoted by differential operator $L=\frac{\partial^2}{\partial x^2}$, R is called linear operator, Nu represents a general non-linear differential operator and $h(x,t)$ is source term. The methodology consists of applying double Laplace transform on both sides of Eq(10)

$$L_xL_t\left[Lu(x,t)+Ru(x,t)+Nu(x,t)=h(x,t)\right] \tag{12}$$

The frist step we applying the differentiation property of double and single Laplace transform we get

$$U(p,s)=\frac{F(s)}{p}+\frac{G(s)}{p^2}-\frac{1}{p^2}L_xL_t\left[Ru(x,t)+Nu(x,t)\right]+\frac{1}{p^2}L_xL_t\left[h(x,t)\right] \tag{13}$$

The second step in Laplace decomposition method is that we represent solution as an infinite series given below

$$u(x,t)=\sum_{n=0}^{\infty}u_n(x,t), \tag{14}$$

The nonlinear operator is decompose as follows

$$Nu=\sum_{n=0}^{\infty}A_n \tag{15}$$

where A_n are Adomian polynomials given by Eq(9). By subsitituting Eq(14) and Eq(15) into Eq(13) we obtain

$$L_xL_t\left(\sum_{n=0}^{\infty}u_n(x,t)\right)=\frac{F(s)}{p}+\frac{G(s)}{p^2}-\frac{1}{p^2}L_xL_t\left[R\sum_{n=0}^{\infty}u_n(x,t)+\sum_{n=0}^{\infty}A_n\right]+\frac{1}{p^2}L_xL_t\left[h(x,t)\right]. \tag{16}$$

Now, applying the inverse double Laplace transform on both sides of Eq(16), we get

$$\left(\sum_{n=0}^{\infty}u_n(x,t)\right)=f(t)+xg(t)-L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[R\sum_{n=0}^{\infty}u_n(x,t)+\sum_{n=0}^{\infty}A_n\right]\right]+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[h(x,t)\right]\right]. \tag{17}$$

On comparing both sides of the Eq(17) we have

$$u_0(x,t)=f(t)+xg(t)+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[h(x,t)\right]\right] \tag{18}$$

$$u_1(x,t)=L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[Ru_0(x,t)+A_0\right]\right] \tag{19}$$

$$u_2(x,t)=L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[Ru_1(x,t)+A_1\right]\right]. \tag{20}$$

In general the recursive relation is given by

$$u_{n+1}(x,t)=-L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[Ru_n(x,t)+A_n\right]\right],\ \ n\geq 0. \tag{21}$$

Now first of all we applying double Laplace transform of the terms on the right hand side of Eq(21) then applying inverse double Laplace transform we get the values of $u_1,u_2,\ldots,u_n$ respectively

2 Applications

To demonstrate the applicability of the above-presented method, for nonlinear partial differential equations, we now consider some examples.

Example 1. We Consider the nonlinear partial differential equation

$$u_{xx}+(u_t)^2+u-u^2=-xe^{-t} \tag{22}$$

with initial conditions

$$u(0,t)=0,\quad u_x(0,t)=e^{-t}. \tag{23}$$

On using double and single laplace transform method we obtain

$$U(p,s)=\frac{1}{p^2(s+1)}+\frac{1}{p^4(s+1)}-\frac{1}{p^2}L_xL_t\left[(u_t)^2\right]+\frac{1}{p^2}L_xL_t\left[u^2\right]-\frac{1}{p^2}L_xL_t\left[u\right]. \tag{24}$$

By taking double inverse Laplace transform, we have

$$u(x,t)=xe^{-t}+\frac{x^3e^{-t}}{6}-L_p^{-1}L_s^{-1}\left(\frac{1}{p^2}L_xL_t\left[(u_t)^2\right]\right)+L_p^{-1}L_s^{-1}\left(\frac{1}{p^2}L_xL_t\left[u^2\right]\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{p^2}L_xL_t\left[u\right]\right). \tag{25}$$

Applying Eq(17) we get

$$\sum_{n=0}^{\infty} u_n(x,t) = xe^{-t} - \frac{x^3e^{-t}}{6} - L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} A_n\right]\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} B_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} u_n\right]\right] \tag{26}$$

where A_n and B_n are Adomian polynomials that represent by

$$(u_t)^2 = \sum_{n=0}^{\infty} A_n,\ \ u^2 = \sum_{n=0}^{\infty} B_n. \tag{27}$$

The beginning terms of Adomian polynomials, which is given by

$$\begin{aligned} &A_0 = (u_0)_t^2, A_1 = 2(u_0)_t(u_1)_t, A_2 = (u_1)_t^2 + 2(u_0)_t(u_2)_t \\ &B_0 = u_0^2, B_1 = 2u_0u_1 B_2 = u_1^2 + 2u_0u_2. \end{aligned} \tag{28}$$

By using Eq(18),Eq(21) and Eq(26) we get

$$\begin{aligned} u_0(x,t) &= xe^{-t} + \frac{x^3e^{-t}}{6} \\ u_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} A_n\right]\right] + L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} B_n\right]\right] - L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty} u_n\right]\right] \end{aligned} \tag{29}$$

Applying above recursive relation, we obtain

$$\begin{aligned} u_1(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[xe^{-t} + \frac{x^3e^{-t}}{6}\right]\right] \\ &= -L_p^{-1}L_s^{-1}\left[\frac{1}{p^4(s+1)} + \frac{1}{p^6(s+1)}\right] \\ &= -\frac{x^3e^{-t}}{6} - \frac{1}{5!}x^5e^{-t} \end{aligned} \tag{30}$$

and

$$\begin{aligned} u_2(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[-\frac{x^3e^{-t}}{6} - \frac{1}{5!}x^5e^{-t}\right]\right] \\ &= -L_p^{-1}L_s^{-1}\left[-\frac{1}{p^6(s+1)} - \frac{1}{p^8(s+1)}\right] \\ &= \frac{1}{5!}x^5e^{-t} + \frac{1}{7!}x^7e^{-t}. \end{aligned} \tag{31}$$

We see that the second term in u_0 and the first terms in u_1 becomes zero, keeping the non noise terms in u_0 obtain the exact solution of Eq(22) as follow

$$u(x,t) = xe^{-t}$$

Example 2. Consider the following nonlinear partial differential equation [6]

$$u_{xx} - uu_{tt} = -x^2e^{-2t} \tag{32}$$

with initial conditions

$$u(0,t) = 0,\ \ u_x(0,t) = e^{-t}. \tag{33}$$

By applying the aforesaid method subject to the initial condition, we have

$$U(p,s)=\frac{1}{p^2(s+1)}-\frac{2}{p^5(s+2)}+\frac{1}{p^2}L_xL_t\left[uu_{tt}\right] \tag{34}$$

On using inverse double Laplace transform

$$u(x,t)=xe^{-t}-\frac{x^4e^{-2t}}{12}+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[uu_{tt}\right]\right] \tag{35}$$

by using Eq(17) we get

$$\sum_{n=0}^{\infty}u_n(x,t)=xe^{-t}-\frac{x^4e^{-2t}}{12}+L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty}A_n\right]\right] \tag{36}$$

where A_n are Adomian polynomials that represent

$$uu_{tt}=\sum_{n=0}^{\infty}A_n.$$

The first few components of Adomian polynomials, are given by

$$\begin{aligned}A_0&=(u_0)(u_0)_{tt}\\A_1&=(u_0)(u_1)_{tt}+(u_1)(u_0)_{tt}\\A_2&=(u_0)(u_2)_{tt}+(u_1)(u_1)_{tt}+(u_2)(u_0)_{tt}\end{aligned}$$

The recursive relation is given below

$$\begin{aligned}u_0(x,t)&=xe^{-t}-\frac{1}{12}x^4e^{-2t}\\u_{n+1}(x,t)&=L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty}A_n\right]\right]\end{aligned} \tag{37}$$

The other components of the solution can easily found by using above recursive relation

$$\begin{aligned}u_1(x,t)&=L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[\sum_{n=0}^{\infty}A_0\right]\right]\\&=L_p^{-1}L_s^{-1}\left[\frac{1}{p^2}L_xL_t\left[x^2e^{-2t}-\frac{5}{12}x^5e^{-3t}+\frac{1}{36}x^8e^{-4t}\right]\right]\\&=\frac{1}{12}x^4e^{-2t}-\frac{5}{12\times 42}x^7e^{-3t}+\frac{1}{36\times 90}x^{10}e^{-4t}\end{aligned} \tag{38}$$

It is obvious that the self-canceling "noise" terms appear between various components. Canceling the second term in u_0 and the first terms in u_1 , keeping the non noise terms in u_0 yields the exact solution of Eq(32) given by

$$u(x,t)=xe^{-t}$$

Example 3. Consider nonlinear partial differential equation [7]

$$u_{xx}-u_xu_{tt}=-x+u \tag{39}$$

with initial conditions

$$u(0,t) = \sin t, \quad u_x(0,t) = 1$$

Applying the double and single Laplace transform we get

$$U(p,s) = \frac{1}{p(s^2+1)} + \frac{1}{p^2 s} - \frac{1}{p^4 s} + \frac{1}{p^2} L_x L_t [u_x u_{tt} + u] \tag{40}$$

The inverse of double Laplace transform implies that

$$u(x,t) = \sin t + x - \frac{x^3}{3!} + L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t [u_x u_{tt} + u] \right] \tag{41}$$

We decompose the solution as an infinite sum given below

$$\sum_{n=0}^{\infty} u_n(x,t) = \sin t + x - \frac{x^3}{3!} + L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t \left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n(x,t) \right] \right] \tag{42}$$

The nonlinear term is handled with the help of Adomian polynomials [7] as

$$u_x u_{tt} = \sum_{n=0}^{\infty} A_n$$

The recursive relation is given below

$$\begin{aligned} u_0(x,t) &= \sin t + x - \frac{x^3}{3!} \\ u_1(x,t) &= L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t \left[\sum_{n=0}^{\infty} A_0 + \sum_{n=0}^{\infty} u_0(x,t) \right] \right] \\ u_{n+1}(x,t) &= L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t \left[\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} u_n(x,t) \right] \right] \end{aligned} \tag{43}$$

The other components of the solutions can be easily found by using above recursive relation

$$\begin{aligned} u_1(x,t) &= L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t [(u_0)_x (u_0)_{tt} + u_0(x,t)] \right] \\ &= L_p^{-1} L_s^{-1} \left[\frac{1}{p^2} L_x L_t \left[\frac{x^2}{2} \sin t + x - \frac{x^3}{3!} \right] \right] \\ &= L_p^{-1} L_s^{-1} \left[\frac{1}{p^5 (s^2+1)} + \frac{1}{p^4 s} - \frac{1}{p^6 s} \right] \\ &= \frac{x^4}{4!} \sin t + \frac{x^3}{3!} - \frac{x^5}{5!} \end{aligned}$$

It is important to recall here that the noise terms appear between the components $u_0(x,t)$ and $u_1(x,t)$, where the noise terms are those pairs of terms that are identical but carrying opposite signs. More precisely, the noise terms $\pm\frac{x^3}{3!}$ between the components $u_0(x,t)$ and $u_1(x,t)$ can be cancelled and the remaining terms of $u_0(x,t)$ still satisfy the equation. Therefore, the exact solution is given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sin t + x$$

Example 4. Consider one dimensional nonlinear wave-like equation [8]

$$u_{tt}=x^2\frac{\partial}{\partial x}\left(u_xu_{xx}\right)-x^2\left(u_{xx}\right)^2-u, \tag{44}$$

with the initial conditions

$$u\left(x,0\right)=0,\ \ u_t\left(x,0\right)=x^2 \tag{45}$$

Taking the double Laplace transform there is (denoted by L_2) on both sides of Eq(44) we obtain

$$s^2U\left(p,s\right)-sU\left(p,0\right)-\frac{\partial U\left(p,0\right)}{\partial t}=L_xL_t\left(x^2\frac{\partial}{\partial x}\left(u_xu_{xx}\right)-x^2\left(u_{xx}\right)^2-u\right). \tag{46}$$

Applying the initial conditions given in Eq(45) , we have

$$U\left(p,s\right)=\frac{2}{p^3s^2}+\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}\left(u_xu_{xx}\right)-x^2\left(u_{xx}\right)^2-u\right). \tag{47}$$

Operating the inverse double Laplace transform on both sides of Eq(47) , we have

$$u(x,t)=x^2t+L_2^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}\left(Nu\right)-x^2\left(Mu\right)-u\right)\right), \tag{48}$$

where

$$Nu=u_xu_{xx}=\sum_{n=0}^{\infty}A_n,\qquad Mu=\left(u_{xx}\right)^2=\sum_{n=0}^{\infty}B_n \tag{49}$$

Using Eq(49) we obtain Adomian's polynomials as follows

$$\begin{aligned}
A_0&=\left(u_0\right)_x\left(u_0\right)_{xx}\\
A_1&=\left(u_0\right)_x\left(u_1\right)_{xx}+\left(u_1\right)_x\left(u_0\right)_{xx}\\
A_2&=\left(u_0\right)_x\left(u_2\right)_{xx}+\left(u_1\right)_x\left(u_1\right)_{xx}+\left(u_2\right)_x\left(u_0\right)_{xx}
\end{aligned}$$

and

$$B_0=\left(u_0\right)_{xx}^2,B_1=2\left(u_0\right)_{xx}\left(u_1\right)_{xx},B_2=\left(u_1\right)_{xx}^2+2\left(u_0\right)_{xx}\left(u_2\right)_{xx}$$

Starting with $u_0(x,t)=x^2t$ and using

$$u_{n+1}(x,t)=L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}\sum_{n=0}^{\infty}A_n\right)\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\sum_{n=0}^{\infty}B_n\right)\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\sum_{n=0}^{\infty}\left(u_n\right)\right). \tag{50}$$

We can obtain

$$\begin{aligned}
u_1(x,t)&=L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}\left(A_0\right)\right)\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\left(B_0\right)\right)\right)-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(u_0\right)\right)\\
&=-L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2t\right)\right)=-L_p^{-1}L_s^{-1}\left(\frac{2}{p^3s^4}\right)=-\frac{x^2t^3}{3!}
\end{aligned} \tag{51}$$

$$
\begin{aligned}
u_2(x,t) &= L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2\frac{\partial}{\partial x}(A_1)\right)\right) - L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(x^2(B_1)\right)\right) - L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t(u_1)\right) \\
&= -L_p^{-1}L_s^{-1}\left(\frac{1}{s^2}L_xL_t\left(-\frac{x^2t^3}{3!}\right)\right) = L_p^{-1}L_s^{-1}\left(\frac{2}{p^3s^6}\right) = \frac{x^2t^5}{5!}
\end{aligned} \tag{52}
$$

$$
u_3(x,t) = -\frac{x^2t^7}{7!}
$$

On using Eq(14), we have

$$
u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \ldots\right)
$$

The exact solution is $u(x,t) = x^2 \sin t$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors have contributed to all parts of the article. All authors read and approved the final manuscript.

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