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THE LAGRANGIAN DENSITY OF $\{123, 234, 456\}$ AND THE TURÁN NUMBER OF ITS EXTENSION

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Abstract

Given a positive integer n and an r-uniform hypergraph F, the *Turán number* $ex(n, F)$ is the maximum number of edges in an F-free r-uniform hypergraph on n vertices. The *Turán density* of F is defined as $\pi(F) = \lim_{n\to\infty} \frac{ex(n,F)}{\binom{n}{r}}$. The *Lagrangian density* of F is $\pi_\lambda(F) = \sup\{r!\lambda(G) : G$ is F-free$\}$, where $\lambda(G)$ is the Lagrangian of G. Sidorenko observed that $\pi(F) \leq \pi_\lambda(F)$, and Pikhurko observed that $\pi(F) = \pi_\lambda(F)$ if every pair of vertices in F is contained in an edge of F. Recently, Lagrangian densities of hypergraphs and Turán numbers of their extensions have been studied actively. For example, in the paper [*A hypergraph Turán theorem via Lagrangians of intersecting families*, J. Combin. Theory Ser. A 120 (2013) 2020–2038], Hefetz and Keevash studied the Lagrangian densitiy of the 3-uniform graph spanned by $\{123, 456\}$ and the Turán number of its extension. In this paper, we show that the Lagrangian density of the 3-uniform graph

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spanned by $\{123, 234, 456\}$ achieves only on K_5^3 . Applying it, we get the Turán number of its extension, and show that the unique extremal hypergraph is the balanced complete 5-partite 3-uniform hypergraph on n vertices.

Keywords: Turán number, hypergraph Lagrangian, Lagrangian density.

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